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Free Field Approach to String Theory on AdS$_3$

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Abstract

We discuss the correlation functions of the $SL(2,\mathbf{C})/SU(2)$ WZW model, or the CFT on the Euclidean AdS_3. We argue that the calculation is reduced to that of a free theory by taking into account the renormalization and integrating out a certain zero-mode, which is an analog of the zero-mode integration in Liouville theory. Based on the resultant free field picture, we give a simple prescription for calculating the correlation functions. The exact two- and three-point functions of generic primary fields are correctly reproduced, including numerical factors. We also obtain some four-point functions of primaries by solving the Knizhnik-Zamolodchikov equation, and verify that our prescription can reproduce them.

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1 Introduction

The $SL(2,\mathbf{R})$ WZW model is the theory of a string propagating on AdS_3 in the presence of non-zero B-field. It offers us a simple example of a string theory on a curved background. Since AdS_3 is closely related to the geometry of black holes in various dimensions, the study of this theory will lead us to a better understanding of the quantum theory of black holes.

The $SL(2,\mathbf{R})$ WZW model has also an application to the AdS/CFT correspondence[1, 2, 3]. It has been conjectured that the theory of gravity on anti-de Sitter spaces has a holographic dual description by a conformal field theory on the boundary. In the three-dimensional case, the $SL(2,\mathbf{R})$ WZW model allows us to analyze the theory of gravity in bulk at a stringy level using the conventional techniques of conformal field theory[4, 5, 6]. It is then believed that there is a correspondence between two CFTs, namely, between the worldsheet theory describing a string on AdS_3 and the CFT on the boundary of AdS_3 which is called the "space-time" CFT.

However, due to the non-compactness of the target space, our knowledge of the $SL(2,\mathbf{R})$ WZW model is far from complete. There have been many works concerning its fundamental properties such as modular invariance, physical spectrum, fusion rules, unitarity and so on. For recent progress on these issues, see [7]-[17].

For actual applications, the Euclidean version of the $SL(2,\mathbf{R})$ WZW model is equally important. From a group theoretical viewpoint, the Euclidean AdS_3 is equivalent to the quotient space $H_3^+ = SL(2,\mathbf{C})/SU(2)$. The string theory on this space is also described by a WZW-like action. In this paper we mainly consider the Euclidean case.

The H_3^+ WZW model has been studied from various approaches. Since the action reduces to a particularly simple form if we use a certain coordinate system, we can analyze the theory from the Lagrangian approach[18, 19, 15]. Alternatively, we can analyze the system based on a free field realization of current algebra[20]-[25], [9, 13, 14, 17]. One can also study the theory based on the symmetry, bootstrap condition[26, 27] or by solving the Knizhnik-Zamolodchikov(KZ) equation[28, 29]. These different approaches are of complementary use to one another, and believed to be mutually consistent. Therefore, in order to obtain a complete understanding of this theory, it is necessary to clarify how these approaches are related to one another.

In this paper, we argue that the calculation of the correlation functions is reduced to that of a free theory. We start from the full Lagrangian and primary fields. Then by integrating out a certain zero-mode and taking into account the renormalization, it is shown that the expression of the correlators becomes that of a free theory. Using the resultant free field picture, it is further shown that we can calculate the correlation functions explicitly and obtain the results which are consistent with other approaches.

This paper is organized as follows. In section 2 we summarize the H_3^+ WZW model. In section 3 we argue how the free field picture emerges, and give a prescription for calculating correlators. Using this prescription we calculate the two- and three-point functions of primary fields, and reproduce the known results correctly, including numerical factors. In section 4 we obtain some four-point functions of primary fields by solving the KZ equation. We see that it can be solved explicitly if we put a certain condition on the $SL(2,\mathbf{C})$ spins of four primaries. Rewriting the solutions in a form manifestly symmetric in four vertices, we find that they can be easily reproduced from our free field prescription. We conclude with a brief discussion in section 5. The definition and some basic properties of the Υ function and the hypergeometric function are summarized in the appendix A and B, respectively.

2 H_3^+ WZW model

The Euclidean AdS_3 is equivalent to the following quotient space:

$$H_3^+ \equiv SL(2,\mathbf{C})/SU(2) \ , \tag{2.1}$$

and the sigma model on this space turns out to be conformal if the NS-NS B-field is suitably turned on. The theory is described by a WZW-like action:

$$S = \frac{k}{2\pi}\int_\Sigma d^2z \mathrm{Tr}[g^{-1}\partial g g^{-1}\bar\partial g] + \frac{ik}{12\pi}\int_{\partial^{-1}\Sigma}\mathrm{Tr}[(g^{-1}dg)^3] \ , \tag{2.2}$$

where g denotes an element of H_3^+ and we use throughout this paper $z = \sigma^1 + i\sigma^2$, $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $d^2z = d\sigma^1 d\sigma^2 = \frac{i}{2}dzd\bar z$. This theory is conventionally called the H_3^+ WZW model. Putting into the above action (2.2) the following parametrization of g

$$g = \begin{pmatrix} e^\phi\gamma\bar\gamma + e^{-\phi} & e^\phi\gamma \\ e^\phi\bar\gamma & e^\phi \end{pmatrix} \tag{2.3}$$

one obtains

$$S = \frac{k}{\pi}\int d^2z[\partial\phi\bar\partial\phi + e^{2\phi}\partial\bar\gamma\bar\partial\gamma] \ . \tag{2.4}$$

As in ordinary WZW models, this theory has a current algebra $SL(2,\mathbf{C}) \times \overline{SL(2,\mathbf{C})}$. However, unlike ordinary WZW theories, the left- and the right-moving currents are complex conjugate to each other. They are associated to the following symmetry of the action (2.2):

$$S[g(z,\bar z)] = S[h(z)g(z,\bar z)h^\dagger(\bar z)] \ , \tag{2.5}$$

and there are conserved currents corresponding to the above symmetry:

$$\begin{aligned}
j &= -k\partial g g^{-1} = \begin{pmatrix} j^3 & -j^+ \\ j^- & -j^3 \end{pmatrix} , & \bar j &= -kg^{-1}\bar\partial g = \begin{pmatrix} \bar j^3 & \bar j^- \\ -\bar j^+ & -\bar j^3 \end{pmatrix} , \\
j^- &= -ke^{2\phi}\partial\bar\gamma \ , & \bar j^- &= -ke^{2\phi}\bar\partial\gamma \ , \\
j^3 &= -k(e^{2\phi}\partial\bar\gamma\gamma - \partial\phi) \ , & \bar j^3 &= -k(e^{2\phi}\bar\partial\gamma\bar\gamma - \bar\partial\phi) \ , \\
j^+ &= -k(e^{2\phi}\partial\bar\gamma\gamma^2 - 2\gamma\partial\phi - \partial\gamma) \ , & \bar j^+ &= -k(e^{2\phi}\bar\partial\gamma\bar\gamma^2 - 2\bar\gamma\bar\partial\phi - \bar\partial\bar\gamma) \ .
\end{aligned} \tag{2.6}$$

The currents j^A satisfy the following OPEs:

$$\begin{aligned} j^3(z)j^\pm(w) &\sim \frac{\pm j^\pm(w)}{z-w} , \\ j^+(z)j^-(w) &\sim \frac{k}{(z-w)^2} - \frac{2j^3(w)}{z-w} , \\ j^3(z)j^3(w) &\sim \frac{-k}{2(z-w)^2} , \end{aligned} \tag{2.7}$$

and similar relations hold also for $\bar{j}^A$.

An important class of operators are the primary fields $\Phi_j(z,x)$. They are characterized by the following OPEs with currents:

$$j^A(z)\Phi_j(w,x) \sim -\frac{D^A\Phi_j(w,x)}{z-w} , \quad \bar{j}^A(\bar{z})\Phi_j(w,x) \sim -\frac{\bar{D}^A\Phi_j(w,x)}{\bar{z}-\bar{w}} , \tag{2.8}$$

with

$$\begin{aligned} D^- &= \partial_x , & \bar{D}^- &= \partial_{\bar{x}} , \\ D^3 &= x\partial_x - j , & \bar{D}^3 &= \bar{x}\partial_{\bar{x}} - j , \\ D^+ &= x^2\partial_x - 2jx , & \bar{D}^+ &= \bar{x}^2\partial_{\bar{x}} - 2j\bar{x} . \end{aligned} \tag{2.9}$$

In our convention they are normalized as

$$\Phi_j(z,x) \equiv (e^\phi|\gamma - x|^2 + e^{-\phi})^{2j} . \tag{2.10}$$

Under the global $SL(2,\mathbf{C}) \times \overline{SL(2,\mathbf{C})}$ transformation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{C}) : \; g \to A^{-1}g(A^{-1})^\dagger,$$

the primary fields transform as

$$A : \; \Phi_j(z,x) \to |cx+d|^{4j}\Phi_j(z,Ax) , \quad Ax = \frac{ax+b}{cx+d}. \tag{2.11}$$

Since j merely labels the second Casimir of $SL(2,\mathbf{C})$, there must be a relation between the primary fields of spin j and $-j-1$. Classically they are related by

$$\Phi_j(z,x) = \frac{2j+1}{\pi}\int d^2y|x-y|^{4j}\Phi_{-j-1}(z,y), \tag{2.12}$$

but the coefficient in the right hand side might get quantum corrections.

As we have seen, the action reduces to a very simple form when written in terms of ϕ,γ and $\bar{\gamma}$. In particular, it is bilinear in $\gamma,\bar{\gamma}$ so that the path-integration over those fields can be carried out rather easily. In some early works[18, 19] the theory was analyzed in the path-integral formalism, using the action (2.4) and the $SL(2,\mathbf{C})$-invariant measure

$$\mathcal{D}g = \mathcal{D}\phi\mathcal{D}(e^\phi\gamma)\mathcal{D}(e^\phi\bar{\gamma}) . \tag{2.13}$$

It was shown that, after a suitable treatment of divergences arising from zero-mode integrals we can obtain finite result for various correlators. Based on this formalism, a recent work[15] has reproduced the two- and three-point functions of primary fields, which are of particular importance in our understanding of the AdS/CFT correspondence.

3 Lagrangian Approach

In [19, 15], the calculation of correlation functions resembled that of a free theory. However, the precise connection to the ordinary free field approach seemed unclear. In this section, we argue that the full interacting theory actually reduces to a free theory by (i) integrating out the zero-mode of ϕ and (ii) taking into account the renormalization. Consequently, the calculation of the correlation functions is simplified and one can easily obtain the results of the two- and three-point functions in [26, 27, 15]. Moreover, it enables us to get the integral representation of higher point functions under some conditions.

3.1 Emergence of Free Theory

Although the action (2.4) written in ϕ, γ and $\bar{\gamma}$ has a simple form, for our purpose we rewrite it further by introducing auxiliary fields β and $\bar{\beta}$. Then the theory is defined by the action

$$\begin{aligned} S &= S_0 + S_{\mathrm{int}} \ , \\ S_0 &= \frac{1}{\pi}\int d^2z\Big(k\partial\phi\bar{\partial}\phi - \beta\bar{\partial}\gamma - \bar{\beta}\partial\bar{\gamma}\Big) \ , \\ S_{\mathrm{int}} &= -\frac{1}{k\pi}\int d^2z\beta\bar{\beta}e^{-2\phi} \ , \end{aligned} \tag{3.1}$$

and the $SL(2)$ invariant measure

$$\mathcal{D}\phi\mathcal{D}(e^{\phi}\gamma)\mathcal{D}(e^{\phi}\bar{\gamma})\mathcal{D}(e^{-\phi}\beta)\mathcal{D}(e^{-\phi}\bar{\beta}) \ . \tag{3.2}$$

With the above action and measure, correlators are given by

$$\langle X\rangle \equiv \int \mathcal{D}\phi\mathcal{D}(e^{\phi}\gamma)\mathcal{D}(e^{\phi}\bar{\gamma})\mathcal{D}(e^{-\phi}\beta)\mathcal{D}(e^{-\phi}\bar{\beta})\exp[-S]\cdot X \ . \tag{3.3}$$

Although the above action seems almost free, it still describes an interacting theory because of the term S_{int}. In terms of the fields in (3.1), the affine symmetry (2.5) is translated into the symmetry under

$$\begin{aligned} \delta\gamma &= \epsilon(\gamma) \ , \quad \delta\bar{\gamma} = -\frac{1}{2}\epsilon'' e^{-2\phi} \ , \quad \delta\phi = -\frac{1}{2}\epsilon' \ , \\ \delta\beta &= -\epsilon'\beta + \frac{k}{2}e^{2\phi}\partial(e^{-2\phi}\epsilon'') \ , \quad \delta\bar{\beta} = 0 \ . \end{aligned} \tag{3.4}$$

Here, $\epsilon(\gamma) = \epsilon_-(z) + \epsilon_3(z)\gamma + \epsilon_+(z)\gamma^2$ and primes denote the derivatives with respect to γ. Similar transformations hold also for $\bar{j}^A$.

In the following we consider the correlation functions of the primary fields Φ_j. For later use, we expand Φ_j in terms of $e^{-2\phi}$:

$$\Phi_j = \Phi_j^{\mathrm{f}}\Big(1 + \mathcal{O}(e^{-2\phi})\Big) \ , \tag{3.5}$$

where

$$\Phi_j^{\mathrm{f}}(z,x) = |x-\gamma|^{4j} e^{2j\phi} \ . \tag{3.6}$$

The correlator of Φ_j is then written as

$$\begin{aligned} &\left\langle \prod_{a=1}^{N} \Phi_{j_a}(z_a, x_a) \right\rangle \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.7) \\ &\qquad \equiv \int \mathcal{D}\phi\mathcal{D}(e^{\phi}\gamma)\mathcal{D}(e^{\phi}\bar\gamma)\mathcal{D}(e^{-\phi}\beta)\mathcal{D}(e^{-\phi}\bar\beta)\exp[-S]\prod_{a=1}^{N}\Phi_{j_a}^{\mathrm{f}}(z_a,x_a)\Big(1+\mathcal{O}(e^{-2\phi})\Big) \ . \end{aligned}$$

From the index theorem for the fields (β,γ) with spin $(1,0)$, the zero-mode part of the measure is

$$d\phi_0 d\gamma_0 d\bar\gamma_0 d^g\beta_0 d^g\bar\beta_0 e^{2(1-g)\phi_0} \tag{3.8}$$

where $\phi_0, \gamma_0, ...$ are the zero-modes of the respective fields and g is the genus of the worldsheet. In this paper, we focus on the case $g=0$, in which γ has one zero-mode and β has none. Hence, the above expression becomes

$$d\phi_0 d^2\gamma_0 e^{2\phi_0}. \tag{3.9}$$

An important point in our argument is that one can first perform the integration over ϕ_0. The ϕ_0 integral in (3.7) becomes

$$\begin{aligned} &\int d\phi_0 e^{2(\Sigma_a j_a+1)\phi_0}\exp\left[e^{-2\phi_0}\int\frac{d^2w}{k\pi}\beta\bar\beta e^{-2\phi_q}\right]\cdot\Big(1+\mathcal{O}(e^{-2\phi})\Big) \\ &= \frac{1}{2}\Gamma(-\Sigma_a j_a - 1)\left[-\int\frac{d^2w}{k\pi}\beta\bar\beta e^{-2\phi_q}\right]^{\sum_a j_a+1}\cdot\Big(1+\mathcal{O}(e^{-2\phi_q})\Big) \ , \end{aligned}$$

where ϕ_q denotes the non-zero mode of ϕ and $\mathcal{O}(e^{-2\phi_q})$ represents the contributions from the higher order terms in $e^{-2\phi_q}$. Since the interaction term S_{int} appears only in the above form, the remaining functional integration reduces to that with respect to the free action S_0.

Next we perform the functional integration over the non-zero modes. The functional determinant coming from the integration over $\beta\gamma$ and $\bar\beta\bar\gamma$ gives a shift of the kinetic term of ϕ:

$$\det{}^{-1}(e^{-\phi_q}\partial e^{2\phi_q}\bar\partial e^{-\phi_q}) = \exp\left[\int d^2z\left(\frac{2}{\pi}\partial\phi\bar\partial\phi + \frac{\phi_q}{4\pi}\sqrt{g}R\right)\right]. \tag{3.10}$$

The resultant expression for the correlation function of primaries is then

$$\left\langle \prod_{a=1}^{N} \Phi_{j_a}(z_a, x_a) \right\rangle \tag{3.11}$$
$$= \frac{1}{2}\Gamma(-\Sigma_a j_a - 1) \int d\gamma_0 d\bar{\gamma}_0 \left\langle \prod_{a=1}^{N} \Phi^{\rm f}_{j_a}(z_a, x_a)\Big(1 + \mathcal{O}(e^{-2\phi})\Big) {S_{\rm int}}^{\Sigma_a j_a + 1} \right\rangle^{\rm f}_{\phi_0=0,\gamma_0,\bar{\gamma}_0} ,$$

where the bracket $\langle A \rangle^{\rm f}_{\phi_0=0,\gamma_0,\bar{\gamma}_0}$ represents the Wick contraction of A using[1]

$$\begin{array}{rclcrcl} \phi(z) &=& \phi_0 + \phi_q(z), & & \langle \phi_q(z)\phi_q(w)\rangle^{\rm f}_{\phi_0,\gamma_0,\bar{\gamma}_0} &=& -b^2 \ln|z-w| \ , \\ \gamma(z) &=& \gamma_0 + \gamma_q(z), & & \langle \beta(z)\gamma_q(w)\rangle^{\rm f}_{\phi_0,\gamma_0,\bar{\gamma}_0} &=& (z-w)^{-1} \ , \end{array} \tag{3.12}$$

with $b^{-2} = k - 2$.

Furthermore, similarly to the discussions in [19, 15], one can show that the $\mathcal{O}(e^{-2\phi})$ terms disappear after the renormalization because of the self-contraction of $e^{\phi(z_a)}$s (at least when the calculation can be carried out). Hence we arrive at the expression

$$\left\langle \prod_{a=1}^{N} \Phi_{j_a}(z_a, x_a) \right\rangle = \frac{1}{2}\Gamma(-\Sigma_a j_a - 1) \int d\gamma_0 d\bar{\gamma}_0 \left\langle \prod_{a=1}^{N} \Phi^{\rm f}_{j_a}(z_a, x_a) {S_{\rm int}}^{\Sigma_a j_a + 1} \right\rangle^{\rm f}_{\phi_0=0,\gamma_0,\bar{\gamma}_0} . \tag{3.13}$$

The right-hand side is nothing but a correlation function in a free theory in which $S_{\rm int}$ plays the role of the screening operator. Note that there is no ambiguity in how to deal with the "screening operator", since we start from a definite Lagrangian.

An anomaly term as in (3.10) may also be obtained by changing the measure (3.2) to

$$\mathcal{D}\phi\mathcal{D}\gamma\mathcal{D}\bar{\gamma}\mathcal{D}\beta\mathcal{D}\bar{\beta} \ . \tag{3.14}$$

Such a term and S_0 then add up to

$$S_{\rm free} = \frac{1}{\pi}\int d^2z \Big[(k-2)\partial\phi\bar{\partial}\phi - \frac{\phi}{4}\sqrt{g}R - \beta\bar{\partial}\gamma - \bar{\beta}\partial\bar{\gamma}\Big] \ . \tag{3.15}$$

The same ϕ_0 dependence as in (3.8) comes from the term $\phi\sqrt{g}R$ and a similar calculation in the above is possible. However, in this approach, the full $SL(2)$ symmetry (3.4) appears subtle: in the full interacting theory, it is difficult to evaluate the Jacobian from (3.2) to (3.14) and Jacobians of the type $\mathcal{D}(e^{\epsilon(\gamma)}\gamma)/\mathcal{D}\gamma$, which are needed to check the invariance under (3.4).[2]

Alternatively, the expression (3.13) can be obtained by starting with the free theory with the action $S_{\rm free}$ and a perturbation term $S_{\rm int}$. In this picture, all fields are free fields from

[1] We omit the terms which arise because of the background metric.

[2] Strictly speaking, one needs to check that the regularization in calculating (3.10) respects the $SL(2)$ symmetry. However, our results indicate that the procedure in the above actually respect it in total.

the beginning. In particular, $\beta, \gamma (\bar\beta, \bar\gamma)$ constitute a holomorphic (anti-holomorphic) bosonic ghost system.

Let us summarize basic properties of this free field system. We concentrate mainly on the holomorphic sector in the following. First, the stress tensor has the following form:

$$T = \beta\partial\gamma - b^{-2}\partial\phi\partial\phi - \partial^2\phi \ . \tag{3.16}$$

The fundamental fields satisfy the OPEs

$$\phi(z)\phi(w) \sim -b^2 \ln|z-w| \ , \quad \beta(z)\gamma(w) \sim \frac{1}{z-w} \ , \tag{3.17}$$

and using them we can construct the standard free field realization of an $SL(2)$ current of level k:

$$\begin{aligned} j^- &= \beta \ , \\ j^3 &= \beta\gamma + b^{-2}\partial\phi \ , \\ j^+ &= \beta\gamma^2 + 2b^{-2}\gamma\partial\phi + k\partial\gamma \ . \end{aligned} \tag{3.18}$$

The primary fields satisfying the OPEs (2.8) with the above current are given by $\Phi^{\rm f}_j$ in (3.6). They have worldsheet conformal weight $h \equiv -b^2 j(j+1)$.

The interaction $S_{\rm int}$ is made of a screening current $\beta\bar\beta e^{-2\phi}$ which has no singular OPEs with j^A up to total derivatives. Hence the incorporation of $S_{\rm int}$ does not spoil the affine $SL(2)$ symmetry perturbatively.

Generic correlators are defined as follows:

$$\left\langle X[\phi,\beta,\gamma,\bar\beta,\bar\gamma]\right\rangle \equiv \int \mathcal{D}\phi\mathcal{D}\gamma\mathcal{D}\bar\gamma\mathcal{D}\beta\mathcal{D}\bar\beta \exp[-S_{\rm free}] \cdot X \exp[-S_{\rm int}] \ . \tag{3.19}$$

Here we would like to note that the $SL(2)$ current algebra of the free CFT is associated to the following symmetry of path-integration:

$$\begin{gathered} \int \mathcal{D}\phi'\mathcal{D}\gamma'\mathcal{D}\bar\gamma'\mathcal{D}\beta'\mathcal{D}\bar\beta' \exp[-S'_{\rm free}] = \int \mathcal{D}\phi\mathcal{D}\gamma\mathcal{D}\bar\gamma\mathcal{D}\beta\mathcal{D}\bar\beta \exp[-S_{\rm free}] \ , \\ \gamma' = \gamma + \epsilon \ , \quad \beta' = \beta - \epsilon'\beta - b^{-2}\epsilon''\partial\phi + \frac{k}{2}\partial\epsilon'' \ , \quad \phi' = \phi - \frac{1}{2}\epsilon' \ , \end{gathered} \tag{3.20}$$

where ϵ is as given in (3.4).[3] However, the above transformation leaves $S_{\rm int}$ invariant only up to terms proportional to the equation of motion. Hence it is difficult to regard this as the symmetry of a theory based on a Lagrangian in an ordinary sense.

Integrating over non-zero modes in (3.19), we obtain

$$\langle X \rangle = \int d\phi_0 d\gamma_0 d\bar\gamma_0 d^g\beta_0 d^g\bar\beta_0 e^{2(1-g)\phi_0} \left\langle X \exp[-S_{\rm int}]\right\rangle^{\rm f}_{\phi_0,\gamma_0,\bar\gamma_0,\beta_0,\bar\beta_0} \ , \tag{3.21}$$

[3]Here, there is a subtlety again in evaluating Jacobians such as $\mathcal{D}(e^{\epsilon(\gamma)}\gamma)/\mathcal{D}\gamma$.

where the bracket $\langle A \rangle_{\phi_0,\cdots}$ represents the Wick contraction of A as before. For correlators of primaries on a sphere, we actually obtain the same expression as (3.13) by substituting $X = \prod_{a=1}^{N} \Phi^{\mathrm{f}}_{j_a}(z_a, x_a)$ and integrating over ϕ_0.

Although we have obtained the same expressions for correlators, the underlying symmetry in the free field approach is different from that of the full interacting theory, and holds only on-shell or perturbatively. Hence it is more appropriate to start from the full treatment if we try to analyze the theory with the full symmetry (3.4) and regard the invariance of correlation functions as originating from the symmetry of the Lagrangian.

The procedure leading to (3.13) may be an analog of the Liouville case discussed by Goulian and Li [30] (see also [31, 32, 33]). However, since the H_3^+ theory is a little more complicated than Liouville theory, we needed to take into account the renormalization in addition to the integral over ϕ_0.

Using the free field prescription obtained in this way and, in particular, the formula (3.13), we would like to obtain the explicit forms of the correlators and analytically continue them in j_a similarly to [34, 35]. Such a continuation may be justified along the same line as in the Liouville case [36, 37]. Indeed, we will see that our correlators are in complete agreement with the exact results obtained by other approaches[26, 27, 15].

For the expression (3.13) to make sense, $\sum_a j_a + 1$ should be a non-negative integer. However, the prefactor $\Gamma(-\sum_a j_a - 1)$ is then divergent. Similar divergences appear also in Liouville theory. In that case, they arise inevitably if we define correlation functions as analytic functions of complex j_a [34, 35]. The situation in our case seems similar and $\sum_a j_a + 1 \in \mathbf{Z}_{\geq 0}$ may be interpreted as a kind of "mass-shell" condition according to the Liouville case [38, 35].

We also have a comment on the validity of the free field approach. It is usually taken to be valid as $\phi \to \infty$, namely, near the boundary of AdS_3, because the interaction term S_{int} is vanishing there. The primaries Φ_j reduce to Φ^{f}_j in that limit. An important consequence of our argument is that the free field approach is more powerful as long as we consider the correlation functions of the primaries.

Hereafter we consider the two- and three-point functions of primary fields on a sphere according to the above prescription.

3.2 Two-Point Function

For the two-point function, the formula (3.13) and the integration over $\gamma_0, \bar{\gamma}_0$ yield

$$\begin{aligned}
&\langle \Phi_{j_1}(z_1, x_1)\Phi_{j_2}(z_2, x_2)\rangle \\
&= \frac{1}{2}\Gamma(-j_1 - j_2 - 1) \int d\gamma_0 d\bar{\gamma}_0 \left\langle \prod_{a=1,2} |\gamma(z_a) - x_a|^{4j_a} e^{2j_a\phi(z_a)} \cdot S_{\mathrm{int}}{}^{j_1+j_2+1} \right\rangle^{\mathrm{f}}_{\phi_0=0,\gamma_0,\bar{\gamma}_0} \\
&= \frac{\pi}{2}\Gamma(-j_1 - j_2 - 1)\Delta(2j_1 + 1)\Delta(2j_2 + 1)\Delta(-2j_1 - 2j_2 - 1)
\end{aligned}$$

$$
\cdot \left\langle |\gamma_{12} - x_{12}|^{4j_1+4j_2+2} e^{2j_1\phi(z_1)} e^{2j_2\phi(z_2)} S_{\rm int}{}^{j_1+j_2+1} \right\rangle_{\phi_0=\gamma_0=\bar\gamma_0=0}
$$

$$
\begin{aligned}
= \ & \frac{\pi}{2}\Gamma(-j_1-j_2-1)\Delta(2j_1+1)\Delta(2j_2+1)\Delta(-j_1-j_2)\Gamma(j_1+j_2+2)^{-2}|x_{12}|^{2j_1+2j_2} \\
& \cdot \left\langle |\gamma_{12}|^{2j_1+2j_2+2} e^{2j_1\phi(z_1)} e^{2j_2\phi(z_2)} \left[-S_{\rm int}\right]^{j_1+j_2+1} \right\rangle^{\rm f}_{\phi_0=\gamma_0=\bar\gamma_0=0} .
\end{aligned}
$$

Here $\gamma_{12} = \gamma(z_1) - \gamma(z_2)$, $x_{12} = x_1 - x_2$ and we have introduced $\Delta(x) \equiv \Gamma(x)/\Gamma(1-x)$. In the second line we have used

$$
\int d^2x |x|^{4j_1} |1-x|^{4j_2} = \pi\Delta(2j_1+1)\Delta(2j_2+1)\Delta(-2j_1-2j_2-1) ,
$$

which will be of repeated use in the following. In the third line we have expanded a binomial $|\gamma_{12} - x_{12}|^{4j_1+4j_2+2}$ and picked up the relevant term. Of course this is a valid operation only when $j_1 + j_2 \in \mathbf{Z}_{\geq 0}$, although it might be justified in generic cases through the "fractional calculus" which was used in [23]. Then we have made the replacement of coefficients

$$
(-)^{j_1+j_2+1}\left[\frac{(2j_1+2j_2+1)!}{(j_1+j_2)!}\right]^2 \Longrightarrow \Delta(-j_1-j_2)\Delta(2j_1+2j_2+2) .
$$

The free CFT correlator is given by a Dotsenko-Fateev integral[39, 40]:

$$
\begin{aligned}
& \frac{1}{\Gamma(j_1+j_2+2)^2}\left\langle |\gamma_{12}|^{2j_1+2j_2+2} e^{2j_1\phi(z_1)} e^{2j_2\phi(z_2)} \left[\int \frac{d^2w}{k\pi}\beta\bar\beta e^{-2\phi}(w)\right]^{j_1+j_2+1} \right\rangle^{\rm f}_{\phi_0=\gamma_0=\bar\gamma_0=0} \\
= \ & |z_{12}|^{2b^2 j_1(j_1+1)+2b^2 j_2(j_2+1)} K(j_1 - \frac{1}{2b^2},\ j_2 - \frac{1}{2b^2},\ 0),
\end{aligned}
$$

where

$$
\begin{aligned}
K(\alpha_1,\alpha_2,\alpha_3) \ & \equiv \ \int \prod_{i=1}^{n} \frac{d^2y_i}{k\pi} |y_i|^{4b^2\alpha_1} |1-y_i|^{4b^2\alpha_2} \prod_{i<j} |y_i - y_j|^{-4b^2} \\
& = \ \frac{[k^{-1}b^{-2b^2}\Delta(b^2)]^n \Upsilon[b]\Upsilon[-2\alpha_1 b]\Upsilon[-2\alpha_2 b]\Upsilon[-2\alpha_3 b]}{\Gamma(n+1)\Upsilon[-(\sum\alpha_i+1)b]\Upsilon[-\alpha_{12}b]\Upsilon[-\alpha_{23}b]\Upsilon[-\alpha_{31}b]} ,
\end{aligned} \tag{3.22}
$$

$$
n = \sum\alpha_i + 1 + b^{-2} , \quad \alpha_{12} = \alpha_1 + \alpha_2 - \alpha_3 , \quad \alpha_{23} = \alpha_2 + \alpha_3 - \alpha_1 , \quad \alpha_{13} = \alpha_1 + \alpha_3 - \alpha_2 .
$$

Here the integral is expressed using the Υ-function, which was introduced in [35, 34] to write down the three-point functions in Liouville theory for generic parameters. The definition and some basic properties of $\Upsilon(x)$ are summarized in the appendix A.

In calculating further, note that the expression (3.22) for $K(\alpha_i)$ has a subtlety at $\alpha_3 \sim 0$, since the enumerator vanishes as $\sim \alpha_3$. Defining the value of $K(\alpha_1, \alpha_2, 0)$ as the limit $\alpha_3 \to 0$, we can easily see that it is supported on the zeroes of the denominator. Hence $K(\alpha_i)$ becomes delta-functional in the limit. Analyzing carefully, the relevant zeroes of the denominator are at $j_1+j_2+1 = 0$ and at $j_1 = j_2$. Then, similarly to [27, 15], the residues are readily calculated and we obtain

$$
\begin{aligned}
&\langle \Phi_{j_1}(z_1,x_1)\Phi_{j_2}(z_2,x_2)\rangle \\
&\qquad = |z_{12}|^{4b^2 j_1(j_1+1)}[A(j_1)\delta^2(x_{12}) i\delta(j_1+j_2+1) + B(j_1)|x_{12}|^{4j_1} i\delta(j_1-j_2)] \ , \\
A(j) &= -\frac{\pi^3}{(2j+1)^2} \ , \\
B(j) &= b^2\pi^2[k^{-1}\Delta(b^2)]^{2j+1}\Delta[-b^2(2j+1)] \ .
\end{aligned}
\tag{3.23}
$$

Comparing this with the result of [15], we see that they agree precisely up to an overall numerical factor. We also find an agreement with [26, 27] by appropriate changes of normalizations of the primaries.

3.3 Three-Point Function

One has to make use of the symmetry of the full theory (3.4) in calculating the three-point function. Otherwise we encounter a free CFT correlator which is given by an unnecessarily complicated integral.

Using the symmetry (3.4) and the transformation property (2.11), one finds

$$
\left\langle \prod_{a=1}^{3} \Phi_{j_a}(z_a, Ax_a) \right\rangle = \prod_{a=1}^{3} |cx_a+d|^{-4j_a} \left\langle \prod_{a=1}^{3} \Phi_{j_a}(z_a, x_a) \right\rangle \ . \tag{3.24}
$$

Hence we can extract out the x_a-dependence in the following way:

$$
\begin{aligned}
\left\langle \prod_{a=1}^{3} \Phi_{j_a}(z_a, x_a) \right\rangle &= \prod_{a<b} |x_{ab}|^{2j_{ab}} D(j_a, z_a) \ , \\
D(j_a, z_a) &= \prod_{a<b} |x_{ab}|^{-2j_{ab}} \left. \left\langle \prod_{a=1}^{3} \Phi_{j_a}(z_a, x_a) \right\rangle \right|_{x_{1,2,3}=0,1,\infty} \ ,
\end{aligned}
\tag{3.25}
$$

where we have used the notation $j_{12} = j_1 + j_2 - j_3$, etc. The remaining part can be calculated in a similar way as in the case of the two-point function:

$$
\begin{aligned}
D(j_a, z_a) &= \left\langle |\gamma(z_1)|^{4j_1} |1-\gamma(z_2)|^{4j_2} \prod_{a=1}^{3} e^{2j_a\phi(z_a)} \right\rangle \\
&= \frac{\pi}{2}\Gamma(-\Sigma j_a - 1)\Delta(2j_1+1)\Delta(2j_2+1)\Delta(-j_{12})\Gamma(\Sigma j_a+2)^{-2} \\
&\quad \cdot \left\langle |\gamma_{12}|^{2\Sigma j_a+2} \prod_{a=1}^{3} e^{2j_a\phi(z_a)} [-S_{\rm int}]^{\Sigma j_a+1} \right\rangle^{\rm f}_{\phi_0=\gamma_0=\bar\gamma_0=0} \\
&= \frac{\pi}{2}\Gamma(-\Sigma j_a - 1)\Delta(2j_1+1)\Delta(2j_2+1)\Delta(-j_{12}) \cdot \prod_{a<b} |z_{ab}|^{-4b^2 j_a j_b} \\
&\quad \cdot \int \prod_{i=1}^{\Sigma j_a+1} \frac{d^2 y_i}{k\pi} \left| \frac{z_{12}}{(y_i-z_1)(y_i-z_2)} \right|^2 \prod_a |z_a - y_i|^{4b^2 j_a} \cdot \prod_{i<j} |y_i - y_j|^{-4b^2} \ .
\end{aligned}
$$

Here we have performed an expansion of a binomial which is valid when $j_1 + j_2 - j_3 \in \mathbf{Z}_{\geq 0}$ and $\sum_a j_a + 1 \in \mathbf{Z}_{\geq 0}$. Then we extract the z_a-dependence in the same way. Looking carefully into the worldsheet $SL(2,\mathbf{C})$ transformation property, one finds

$$D(j_a, z_a) = D(j_a) \prod_{a<b} |z_{ab}|^{-2h_{ab}} \ , \qquad \text{where} \ \ h_{12} = h_1 + h_2 - h_3, \ \text{etc.} \tag{3.26}$$

The numerical factor $D(j_a)$ is again given by a Dotsenko-Fateev integral:

$$D(j_a) = \frac{\pi}{2}\Gamma(-\Sigma j_a - 1)\Delta(2j_1+1)\Delta(2j_2+1)\Delta(-j_{12}) \cdot K(j_1 - \frac{1}{2b^2}, j_2 - \frac{1}{2b^2}, j_3) \ .$$

Summarizing, the three-point function is given by

$$\begin{aligned} \left\langle \prod_{a=1}^{3} \Phi_{j_a}(z_a, x_a) \right\rangle &= D(j_a) \prod_{a<b} |z_{ab}|^{-2h_{ab}} |x_{ab}|^{2j_{ab}} \ , \\ D(j_a) &= \frac{b^2\pi}{2} \frac{[k^{-1}b^{-2b^2}\Delta(b^2)]^{\Sigma j_a+1} \Upsilon[b]\Upsilon[-2j_1 b]\Upsilon[-2j_2 b]\Upsilon[-2j_3 b]}{\Upsilon[-(\Sigma j_a+1)b]\Upsilon[-j_{12}b]\Upsilon[-j_{13}b]\Upsilon[-j_{23}b]} \ . \end{aligned} \tag{3.27}$$

This is again in precise agreement with the known results.

Before concluding the analysis, we would like to note that our two- and three-point functions are indeed consistent with the following symmetry of primary fields:

$$\begin{aligned} \Phi_j(z,x) &= R(j) \int d^2y |x-y|^{4j} \Phi_{-j-1}(z,y) \ , \\ R(j) &= -\frac{(2j+1)^2 b^2}{\pi} \Delta[-(2j+1)b^2] \left[k^{-1}\Delta(b^2)\right]^{2j+1} \ . \end{aligned} \tag{3.28}$$

This might be rather surprising since the fields $\Phi_j^{\rm f}$ (3.6) in (3.13) do not have such symmetry. However, this is understood as a non-trivial check that the procedure in subsection 3.1 respected the original $SL(2)$ symmetry.

3.4 Four-Point Function and Beyond

It is straightforward to write down an integral formula for n-point functions of primary fields, though we shall not do this here. However, we would like to emphasize that our prescription is powerful enough to calculate higher point correlation functions, including numerical factors. Note also that, in obtaining an $SL(2,\mathbf{C})$-invariant (in the sense of space-time) result, the integration over the zero-modes of $\phi, \gamma, \bar{\gamma}$ is of crucial importance.

4 Knizhnik-Zamolodchikov Equation

We believe that our prescription proposed in the previous section works also for four-point functions. To see this, we would like to obtain some four-point functions of primary fields from a different approach: by solving the Knizhnik-Zamolodchikov(KZ) equation. Then we will see that the solutions can be reproduced correctly and easily from our prescription.

To begin with, we shall give an introduction to the KZ equation. In CFTs with a current algebra, the stress tensor can be constructed as a bilinear of the current. In the theory under consideration, the stress tensor T and the $SL(2,\mathbf{R})$ current j^A are related in the following way:

$$T = b^2\eta_{AB} :j^A j^B:= \frac{b^2}{2}(:j^+j^-: + :j^-j^+: -2:j^3j^3:) \ . \tag{4.1}$$

Recall that the primary fields Φ_j are characterized by the following OPEs:

$$\begin{aligned} T(z)\Phi_j(w,x) &\sim \frac{h\Phi_j(w,x)}{(z-w)^2} + \frac{\partial_w\Phi_j(w,x)}{z-w} \ , \\ j^A(z)\Phi_j(w,x) &\sim -\frac{D^A\Phi(w,x)}{z-w} \ . \end{aligned}$$

where D^A are given in (2.9). Then if we define the action of the modes of T and j^A on primary fields by

$$\begin{aligned} L_n\Phi_j(w,x) &\equiv \oint \frac{dz}{2\pi i} T(z)\Phi_j(w,x)(z-w)^{n+1} \ , \\ j^A_n\Phi(w,x) &\equiv \oint \frac{dz}{2\pi i} j^A(z)\Phi_j(w,x)(z-w)^n \ , \end{aligned}$$

we find $L_{n>0}\Phi_j(z,x) = j^A_{n>0}\Phi_j(z,x) = 0$. We further find

$$L_{-1}\Phi_j(z,x) = 2b^2\eta_{AB} j^A_{-1} j^B_0 \Phi_j(z,x) \tag{4.2}$$

by rewriting (4.1) in modes and multiplying it on a primary field. Putting the above relation into a correlation function and deforming the contour, one obtains

$$\partial_{z_a}\left\langle \prod_c \Phi_{j_c}(z_c,x_c)\right\rangle = 2b^2\eta_{AB}\sum_{b(\neq a)} \frac{D^A_{[a]}D^B_{[b]}}{z_{ab}} \left\langle \prod_c \Phi_{j_c}(z_c,x_c)\right\rangle , \tag{4.3}$$

where $D^A_{[a]}$ are meant to act on the a-th vertex. Thus we obtain the KZ equation:

$$\left[\partial_{z_a} - b^2 \sum_{b(\neq a)} z_{ab}^{-1}\mathcal{L}_{ab}\right]\left\langle \prod_c \Phi_{j_c}(z_c,x_c)\right\rangle = 0 \ , \tag{4.4}$$

$$\mathcal{L}_{ab} = x_{ab}^2\partial_{x_a}\partial_{x_b} - 2x_{ab}(j_a\partial_{x_b} - j_b\partial_{x_a}) - 2j_aj_b \ .$$

Let us focus on the four-point function. Unlike the two- and three-point functions, their dependence on z_a and x_a are not determined completely from the (worldsheet and space-time) $SL(2,\mathbf{C})$ invariance alone. Generically the four-point function can be written in the following form:

$$\left\langle \prod_{a=1}^4 \Phi_{j_a}(z_a,x_a)\right\rangle = \prod_{a<b}^4 |z_{ab}|^{2b^2\mu_{ab}}|x_{ab}|^{2\lambda_{ab}}F(z,x) \ , \tag{4.5}$$

where z and x are two cross-ratios

$$z \equiv \frac{z_{41}z_{23}}{z_{43}z_{21}} \ , \quad x \equiv \frac{x_{41}x_{23}}{x_{43}x_{21}} \ , \tag{4.6}$$

and the parameters μ_{ab}, λ_{ab} are functions of j_a subject to the following relations:

$$\sum_{b(\neq a)} \lambda_{ab} = 2j_a \ , \quad \sum_{b(\neq a)} \mu_{ab} = 2j_a(j_a+1) \ . \tag{4.7}$$

Rewriting the KZ equation as a differential equation for $F(z,x)$, one obtains

$$\left[\frac{1}{b^2}\frac{\partial}{\partial z} + \frac{\mathcal{A}}{z} + \frac{\mathcal{B}}{z-1}\right] F \ = \ 0 \ ,$$

where

$$\begin{aligned}
\mathcal{A} &= x^2(1-x)\partial_x^2 - \left\{x^2 + (\lambda_{42}+\lambda_{31})x + (\lambda_{43}+\lambda_{21})x(1-x)\right\}\partial_x \\
&\quad +\mu_{41} - 2j_1j_4 - \lambda_{41} + \lambda_{12}\lambda_{24} + \lambda_{13}\lambda_{34} + \frac{\lambda_{13}\lambda_{24}}{1-x} + (1-x)\lambda_{12}\lambda_{34} \ , \\
\mathcal{B} &= x(1-x)^2\partial_x^2 + \left\{(1-x)^2 + (\lambda_{41}+\lambda_{32})(1-x) + (\lambda_{43}+\lambda_{21})x(1-x)\right\}\partial_x \\
&\quad +\mu_{42} - 2j_2j_4 - \lambda_{42} + \lambda_{21}\lambda_{14} + \lambda_{23}\lambda_{34} + \frac{\lambda_{14}\lambda_{23}}{x} + x\lambda_{12}\lambda_{34} \ .
\end{aligned}$$

A suitable choice of λ_{ab} and μ_{ab} leads to a further simplification of the equation. If we put

$$\begin{aligned}
\lambda_{12} &= j_1+j_2-j_3+j_4 \ , & \mu_{12} &= \Delta_1+\Delta_2-\Delta_3+\Delta_4+2j_1j_4+2j_2j_4 \ , \\
\lambda_{13} &= j_1-j_2+j_3-j_4 \ , & \mu_{13} &= \Delta_1-\Delta_2+\Delta_3-\Delta_4-2j_2j_4 \ , \\
\lambda_{14} &= 0 \ , & \mu_{14} &= -2j_1j_4 \ , \\
\lambda_{23} &= -j_1+j_2+j_3-j_4 \ , & \mu_{23} &= -\Delta_1+\Delta_2+\Delta_3-\Delta_4-2j_1j_4 \ , \\
\lambda_{24} &= 0 \ , & \mu_{24} &= -2j_2j_4 \ , \\
\lambda_{34} &= 2j_4 \ , & \mu_{34} &= 2\Delta_4+2j_1j_4+2j_2j_4 \ ,
\end{aligned} \tag{4.8}$$

where $\Delta_a = j_a(j_a+1)$, the KZ equation becomes

$$\begin{aligned}
0 &= \left[\frac{1}{b^2}\frac{\partial}{\partial z} + \frac{xP_0}{z} + \frac{(1-x)P_1}{z-1}\right] F \ , \\
P_i &= x(1-x)\partial_x^2 + \left[\gamma_i - (1+\alpha+\beta)x\right]\partial_x - \alpha\beta \ ,
\end{aligned} \tag{4.9}$$

$$\alpha = -2j_4 \ , \quad \beta = -j_1-j_2+j_3-j_4 \ , \quad \gamma_0 = -2j_1-2j_4 \ , \quad \gamma_1 = 1-j_1+j_2+j_3-j_4 \ .$$

We can see that P_i in the above are nothing but the hypergeometric differentials. Hence it is expected that, under certain conditions on j_a, the solution can be written down explicitly using hypergeometric functions. In the following we shall give some examples in which the KZ equation can be solved rather easily.

4.1 $\sum_a j_a = -1$

The simplest example is the case $\sum_a j_a = -1$. Since $\gamma_0 = \gamma_1$ and the two hypergeometric differentials coincide in this case, one can find a solution with $F(z,x)$ independent of z. The four-point function is therefore given by

$$\left\langle \prod_{a=1}^{4} \Phi_{j_a}(z_a, x_a) \right\rangle = |x_{12}|^{2(j_1+j_2-j_3+j_4)}|x_{13}|^{2(j_1-j_2+j_3-j_4)}|x_{23}|^{2(-j_1+j_2+j_3-j_4)}|x_{34}|^{4j_4} \cdot \prod_{a<b}^{4} |z_{ab}|^{-4b^2 j_a j_b} \cdot f(x) \ , \tag{4.10}$$

where $f(x)$ satisfies the hypergeometric differential equation $P_0 f(x) = 0$. Recalling that $f(x)$ must be real and single-valued if j_a are all real, one obtains

$$\begin{aligned} f(x) &= C \int d^2t |t|^{2\beta-2}|1-t|^{2\gamma_0-2\beta-2}|1-tx|^{-2\alpha} \\ &= C \int d^2t |t|^{4j_3}|1-t|^{4j_2}|1-tx|^{4j_4} \ , \end{aligned} \tag{4.11}$$

where C is a constant. Indeed, as explained in the appendix B, the above $f(x)$ can be understood as a "square" of the hypergeometric function, which is obviously real and single-valued when j_a are all real. The explicit form of $f(x)$ in terms of the hypergeometric function is given through the formula (B.2).

Making a change of the integration variable from t to x_0 defined by

$$t = \frac{x_{03}x_{21}}{x_{01}x_{23}} \ ,$$

the solution can be recast in a form which is manifestly symmetric in four vertices:

$$\left\langle \prod_{a=1}^{4} \Phi_{j_a}(z_a, x_a) \right\rangle = C(j_a) \prod_{a<b}^{4} |z_{ab}|^{-4b^2 j_a j_b} \cdot \int d^2x_0 \prod_a |x_{0a}|^{4j_a} \ . \tag{4.12}$$

4.2 $\sum_a j_a = 0$

The next simplest is the case $\sum_a j_a = 0$ or $\gamma_1 = \gamma_0 + 1$. In this case the two hypergeometric functions $F(\alpha,\beta,\gamma_i;x)$ have a particularly remarkable property: the differentials xP_0 and $(1-x)P_1$ are linearly realized on them. This is due to the following recursion relation of the hypergeometric functions:

$$\begin{aligned} \gamma(1-x)\frac{\partial}{\partial x}F(\alpha,\beta,\gamma;x) &= (\gamma-\alpha)(\gamma-\beta)F(\alpha,\beta,\gamma+1;x) + \gamma(\alpha+\beta-\gamma)F(\alpha,\beta,\gamma;x) \ , \\ x\frac{\partial}{\partial x}F(\alpha,\beta,\gamma+1;x) &= \gamma\left\{F(\alpha,\beta,\gamma;x) - F(\alpha,\beta,\gamma+1;x)\right\} \ . \end{aligned} \tag{4.13}$$

Therefore the solution to the KZ equation can be found as a linear combination of the two hypergeometric functions $F(\alpha,\beta,\gamma_i;x)$,

$$F(z,x) \sim (z-1)g_0(z)F(\alpha,\beta,\gamma_0;x) + zg_1(z)F(\alpha,\beta,\gamma_1;x) \ , \tag{4.14}$$

or of another solutions to $P_i f_i(x) = 0$. Here "$\sim$" means that we forget about the dependence on $\bar{z}$ and $\bar{x}$ for the time being.

Putting the above ansatz into the KZ equation, we obtain

$$
\begin{aligned}
\frac{\partial}{b^2 \partial z}[(z-1)g_0] &= \gamma_0 g_1 - (\alpha+\beta-\gamma_0)g_0 \ , \\
\frac{\partial}{b^2 \partial z}[z g_1] &= -\gamma_0 g_1 - \frac{(\gamma_0-\alpha)(\gamma_0-\beta)}{\gamma_0} g_0 \ .
\end{aligned}
$$

The previous recursion relation can be utilized here again, and we obtain a solution to the KZ equation which has the following form:

$$
\begin{aligned}
F(z,x) \ \sim \ & \gamma_0(1+b^2\gamma_0)(1-z)F(1+b^2\alpha, 1+b^2\beta, 1+b^2\gamma_0; z)F(\alpha,\beta,\gamma_0;x) \\
& + b^2(\gamma_0-\alpha)(\gamma_0-\beta) z F(1+b^2\alpha, 1+b^2\beta, 2+b^2\gamma_0; z)F(\alpha,\beta,\gamma_0+1;x)
\end{aligned} \tag{4.15}
$$

Since there are two choices for $g_i(z)$ and also for the solutions to $P_i f_i(x) = 0$, we have three other independent solutions to the KZ equation. However, as we shall see in a moment, their explicit form are not important in constructing the four-point function in our case of the H_3^+ WZW model.

Now we try to rewrite the above solution using the contour integral expression for the hypergeometric function, so that the permutation symmetry among four vertices becomes manifest. Writing $F(z,x)$ as

$$
\begin{aligned}
F(z,x) &= \gamma_0(1+b^2\gamma_0)(1-z)G_0(z)F_0(x) + b^2(\gamma_0-\alpha)(\gamma_0-\beta) z G_1(z) F_1(x) \ , \\
G_0(z) &= \frac{1}{1-e^{2\pi i(b^2\beta+1)}} \frac{\Gamma(b^2\gamma_0+1)}{\Gamma(b^2\beta+1)\Gamma(b^2(\gamma_0-\beta))} \oint dt t^{b^2\beta}(1-t)^{b^2(\gamma_0-\beta)-1}(1-tx)^{-b^2\alpha-1} \ , \\
G_1(z) &= \frac{1}{1-e^{2\pi i(b^2\beta+1)}} \frac{\Gamma(b^2\gamma_0+2)}{\Gamma(b^2\beta+1)\Gamma(b^2(\gamma_0-\beta)+1)} \oint dt t^{b^2\beta}(1-t)^{b^2(\gamma_0-\beta)}(1-tx)^{-b^2\alpha-1}, \\
F_0(x) &= \frac{1}{1-e^{2\pi i\beta}} \frac{\Gamma(\gamma_0)}{\Gamma(\beta)\Gamma(\gamma_0-\beta)} \oint ds s^{\beta-1}(1-s)^{\gamma_0-\beta-1}(1-sx)^{-\alpha} \ , \\
F_1(x) &= \frac{1}{1-e^{2\pi i\beta}} \frac{\Gamma(\gamma_0+1)}{\Gamma(\beta)\Gamma(\gamma_0-\beta+1)} \oint ds s^{\beta-1}(1-s)^{\gamma_0-\beta}(1-sx)^{-\alpha}
\end{aligned} \tag{4.16}
$$

and making the change of variables from s, t to x_0, z_0 which are defined by

$$
s = \frac{x_{03}x_{21}}{x_{01}x_{23}} \ , \quad t = \frac{z_{03}z_{21}}{z_{01}z_{23}} \ ,
$$

we can rewrite $F(z,x)$ in the following form:

$$
\prod_{a<b}^{4} {z_{ab}}^{b^2\mu_{ab}} {x_{ab}}^{\lambda_{ab}} F(z,x) \sim \prod_{a<b}^{4} {z_{ab}}^{-2b^2 j_a j_b} \oint dz_0 dx_0 \prod_a z_{0a}^{2b^2 j_a} x_{0a}^{2j_a} \cdot \sum_a \frac{j_a}{z_{0a}x_{0a}} \ . \tag{4.17}
$$

Strictly speaking, we have to specify the integration contour, but the choice of contours becomes unimportant after all, because we shall take the "square" of the above expression.

The above expression is obviously symmetric in four vertices, and by taking its naive square we obtain the four-point function:

$$\left\langle \prod_{a=1}^{4} \Phi_{j_a}(z_a,x_a) \right\rangle = C(j_a) \prod_{a<b} |z_{ab}|^{-4b^2 j_a j_b} \int d^2 z_0 d^2 x_0 \prod_a |z_{0a}|^{4b^2 j_a} |x_{0a}|^{4j_a} \cdot \left| \sum_a \frac{j_a}{z_{0a} x_{0a}} \right|^2 . \quad (4.18)$$

The other solutions to the KZ equation can be obtained by starting from (4.16) with α and β in $F_{0,1}(x)$ exchanged each other. Proceeding in a similar way we obtain

$$\begin{aligned} z_{ab}^{b^2\mu_{ab}} x_{ab}^{\lambda_{ab}} F(z,x) \quad \sim \quad & z_{ab}^{-2b^2 j_a j_b} \oint dz_0 dx_0 \prod_a z_{0a}^{2b^2 j_a} x_{0a}^{2j_a} \\ & \cdot \left\{ (x_{01}^{-1} - x_{04}^{-1})(z_{02}^{-1} - z_{04}^{-1}) - (x_{02}^{-1} - x_{04}^{-1})(z_{01}^{-1} - z_{04}^{-1}) \right\} . \end{aligned}$$

By exchanging four vertices we obtain different solutions, but the number of independent solutions is three. By summing over their squares with an appropriate weight, it might be possible to construct a quantity satisfying consistency conditions such as the single-valuedness, symmetry between four vertices, etc. However, it cannot reduce to the known three-point function in the limit $j_4 \to 0$ as one can see by a comparison with the free field result. In other words, of all the solutions to the KZ equations, only one particular solution given in (4.18) is picked up from the full consistency requirements.

4.3 Comparison with Free Field Approach

According to our prescription based on free fields, the four-point function can be calculated by the formula (3.13) with $N = 4$. If we just neglect the factor, by simple calculation we see that the remaining part coincides precisely with the solutions to the KZ equation, (4.12) and (4.18). This again confirms the validity of our prescription. Note that the integration variables x_0 and z_0 in (4.12) and (4.18) correspond to the zero-mode of γ and the position of the screening current $\beta\bar{\beta}e^{-2\phi}$, respectively.

We can write down the integral formula for more generic four-point functions. If $\sum_a j_a + 1$ is a non-negative integer, the four-point function is expressed in terms of a free field correlator with $\sum_a j_a + 1$ insertions of S_{int}:

$$\begin{aligned} \left\langle \prod_{a=1}^{4} \Phi_{j_a}(z_a,x_a) \right\rangle \;&=\; \frac{1}{2}\Gamma(-\Sigma j_a - 1)(-k\pi)^{-\Sigma j_a - 1} \prod_{a<b} |x_{ab}|^{2\lambda_{ab}} |z_{ab}|^{2b^2\mu_{ab}} |f(z,x)|^2 , \\ f(z,x) \;&=\; \frac{\Gamma(j_1 - j_2 + j_3 - j_4 + 1)}{\Gamma(-2j_2)\Gamma(-2j_4)} \int d\gamma \cdot \gamma_0^{2j_1} \\ & \quad \cdot \int_0^1 ds dt \delta(s+t-1) s^{-2j_2-1} t^{-2j_4-1} (\gamma_0 - s - tx)^{-j_1+j_2-j_3+j_4-1} \\ & \quad \cdot \int \prod_{i=1}^{\Sigma j_a + 1} dy_i \left(\frac{s}{y_i - 1} + \frac{tz}{y_i - z} \right) y_i^{2b^2 j_1 - 1} (y_i - 1)^{2b^2 j_2} (y_i - z)^{2b^2 j_4} \prod_{i<j} y_{ij}^{-2b^2} . \end{aligned} \quad (4.19)$$

In this paper, we have argued that the calculation of the correlators in the H_3^+ WZW model is reduced to that of a free theory. The point is the integration over the zero-mode of ϕ and the renormalization.

Since the free CFT used in the calculation has been deduced from the full interacting theory, the results are expected to be exact. Indeed, we have succeeded in reproducing the known exact results for two- and three-point functions. Moreover, our prescription allows us to calculate higher point correlation functions including numerical factors, at least under some conditions. We have actually seen that the four-point functions which are obtained by solving the KZ equation can also be reproduced easily in our formalism. So we are now confident in saying that our prescription works also in calculating other correlation functions.

In comparing the solutions to the KZ equation with our free field results, we can identify one of the integration variables in the former with the zero-modes of $\gamma, \bar{\gamma}$ in the latter. This means that we have to deal with the zero-modes of the fields appropriately in calculating correlators even if we are in the free field approach.

One advantage of using the free field picture may be that we can construct various vertices easily. For example, it is difficult to construct vertices for spectral-flowed representations unless we use some free field realization of the current algebra. The correlators containing such vertices may also be calculated in our approach, but we should move to another system of free fields, since the ϕ, β, γ system is not best-suited for constructing them.

Some comments are in order here with regard to the relation to the other free field approaches. In contrast with our prescription, the integration over zero-modes of $\gamma, \bar{\gamma}$ does not appear explicitly in most of the works. In [23] a similar free field formula for correlation functions with some rational spins was given. (See also [25].) In the case of four-point function, their formula becomes

$$
\begin{aligned}
&\langle j_3 | \, \Phi_{j_2}(1,1)\Phi_{j_4}(z,x) \, | j_1 \rangle \\
&\quad \sim \ \langle \Phi_{j_3}(\infty,\infty)\Phi_{j_2}(1,1)\Phi_{j_4}(z,x)\Phi_{j_1}(0,0)\rangle \\
&\quad \sim \ \left\langle V_{\mathcal{B}} e^{-2j_3\phi}(z_3) \prod_{a=1,2,4} |\gamma(z_a) - x_a|^{4j_a} e^{2j_a\phi(z_a)} {S_{\rm int}}^{j_1+j_2-j_3+j_4} \right\rangle_{\phi_0=\gamma_0=\bar{\gamma}_0=0} , \\
&\qquad V_{\mathcal{B}} = e^{-2\phi}\delta^2(\gamma) \ , \quad z_{1,2,3,4} = (0,1,\infty,z) \ , \quad x_{1,2,3,4} = (0,1,\infty,x) \ .
\end{aligned}
$$

where the definition of primaries are adapted to our convention, and the operator $\delta(\gamma)$ is defined through the bosonization of the $\beta\gamma$ system. We have also limited to the case where the charge can be screened without the insertion of the "second" screening operators $(\beta\bar{\beta}e^{-2\phi})^{k-2}$. Roughly speaking, the above formula is related to our prescription by the conversion $j_4 \to -j_4 - 1$ by means of (3.28). Namely,

$$\langle j_3 | \, \Phi_{j_2}(1,1)\Phi_{j_4}(z,x) \, | j_1 \rangle$$

$$\sim \quad R(j_3) \cdot \int d\gamma_0 d\bar{\gamma}_0 d^2y |x_4 - y|^{4j_3}$$
$$\cdot \left\langle |\gamma(z_3) - y|^{-4j_3-4} e^{-2(j_3+1)\phi(z_3)} \cdot \prod_{a=1,2,4} |\gamma(z_a) - x_a|^{4j_a} e^{2j_a\phi(z_a)} \cdot S_{\mathrm{int}}{}^{j_1+j_2-j_3+j_4} \right\rangle_{\phi_0=0,\gamma_0,\bar{\gamma}_0} ,$$
$$z_{1,2,3,4} = (0, 1, \infty, z) \ , \quad x_{1,2,3,4} = (0, 1, \infty, x) \ .$$

Integrating over y one obtains a delta function for $\gamma(z_3)$, which cancels with the integral over $\gamma_0, \bar{\gamma}_0$. Though we shall not go into details any further, in this way we might obtain a formula without the integration over zero-modes of $\gamma, \bar{\gamma}$. Note also that, the case $j_1+j_2-j_3+j_4 \in \mathbf{Z}_{\geq 0}$ is another special case in which the KZ equation can be solved relatively easily, since β in (4.9) is a negative integer and the hypergeometric differential equations then have a polynomial solution.

As future directions along this work, it would be interesting to study the singular behavior of the four-point function when z approaches x. In some recent work it is identified with the singularity involving a "long string" with a unit winding number[42]. It is also known that, in the $SU(2)$ WZW model the KZ equation for four-point function (written in terms of x_a) is related to the one for five-point function in minimal models[41]. As was discussed in [21, 24], similar relation should also hold in the $SL(2)$ case.

We also expect that our formalism is applicable to the study of the AdS/CFT correspondence. For example, the Virasoro central charge of the space-time CFT might be calculable from our approach. It would be interesting to obtain its precise value and consider why it has to be quantized. Though the quantization is considered to be a non-perturbative effect as has been discussed in [6], we might be able to explain it from our approach.

Finally, we would like to note the similarity between our discussion and that for Liouville theory. In particular, in Liouville theory or the two-dimensional string theory, the N-point correlators can be obtained when the moduli are integrated out [38]. In our case, such correlators correspond to the correlators of the "space-time" CFT. Our discussion in this paper suggests that the correlators of the space-time CFT may be analyzed using the techniques developed for Liouville theory.

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A Υ-FUNCTION

Here we summarize the definition and some basic properties of the Υ function. $\Upsilon(x)$ is defined by

$$\ln \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[\left(\frac{Q}{2} - x \right) \frac{t}{2} \right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] . \tag{A.1}$$

with $Q = b + b^{-1}$. The integral converges in the strip $0 < \mathrm{Re}\, x < Q$. For other x it is defined through the functional relations

$$\begin{aligned} \Upsilon(x+b) &= \Delta(bx) b^{1-2bx} \Upsilon(x) , \\ \Upsilon(x+b^{-1}) &= \Delta(x/b) b^{2x/b-1} \Upsilon(x) , \\ \Upsilon(Q-x) &= \Upsilon(x) . \end{aligned} \tag{A.2}$$

From these relations one finds that $\Upsilon(x)$ has zeros at

$$x = -mb - nb^{-1} , \quad (m+1)b + (n+1)b^{-1} , \quad m, n \in \mathbf{Z}_{\geq 0} . \tag{A.3}$$

The following formula is also useful:

$$\Upsilon'(-mb) = (-1)^m b^{2m} \Upsilon[(m+1)b] \Gamma(m+1)^2 , \tag{A.4}$$

and, as a special case, we have $\Upsilon'(0) = \Upsilon(b)$.

B HYPERGEOMETRIC FUNCTION

In this paper, we have mainly used the following definition of the Hypergeometric function $F(\alpha, \beta, \gamma; x)$:

$$F(\alpha, \beta, \gamma; x) = \frac{1}{1 - e^{2\pi i \beta}} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \oint_{0^+1^+} dt t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tx)^{-\alpha} . \tag{B.1}$$

Here 0^+1^+ means the integration contour encircling 0 and 1 counterclockwise and once for each. It is a solution to the following differential equation

$$\left[x(1-x) \frac{d^2}{dx^2} + [\gamma - (1+\alpha+\beta)x] \frac{d}{dx} - \alpha\beta \right] F(x) = 0 ,$$

with $F(0) = 1$. The other independent solution to the above equation is given by

$$x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) .$$

There is a relation between a naive "square" of hypergeometric functions and their actual squares:

$$
\begin{aligned}
&\int d^2t|t|^{2\beta-2}|1-t|^{2\gamma-2\beta-2}|1-tx|^{-2\alpha} \\
&= \pi\Delta(\beta)\Delta(\gamma-\beta)\Delta(1-\gamma)F(\alpha,\beta,\gamma;x)F(\alpha,\beta,\gamma;\bar{x}) \\
&\quad +\pi\Delta(\gamma-1)\Delta(1-\alpha)\Delta(1+\alpha-\gamma)|x|^{2-2\gamma} \\
&\qquad \cdot F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;\bar{x})\ . \qquad \text{(B.2)}
\end{aligned}
$$

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